Sharp Spectral Asymptotics for 2-dimensional Schrödinger operator with a strong but degenerating magnetic field. II

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Abstract

I consider the same operator as in part I [Ivr10] assuming however that $\mu \ge Ch^{-1}$ and V is replaced by $(2l+1)\mu hF + W$ with $l \in \mathbb{Z}^+$. Under some non-degeneracy conditions I recover remainder estimates up to $O\bigl(\mu^{-\frac{1}{\nu}}h^{-1}+1\bigr)$ but now case $\mu \ge Ch^{-\nu}$ is no more forbidden and the principal part is of magnitude μh^{-1}.

6 Modified V. I. $\mu \le \epsilon h^{-\nu}$

6.1 Introduction

This paper is a continuation of [Ivr10] which is considered as Part I. I consider spectral asymptotics of the magnetic Schrödinger operator

$$A = \frac{1}{2}\Bigl(\sum_{j,k} P_j g^{jk}(x) P_k - V\Bigr), \qquad P_j = D_j - \mu V_j, \quad V = (2l+1)\mu hF + W \tag{6.1}$$

where g^{jk}, V_j, W are smooth real-valued functions of $x \in \mathbb{R}^2$, $l \in \mathbb{Z}^+$ (i.e. $l = 0, 1, \dots$) and (g^{jk}) is positive-definite matrix, $0 < h \ll 1$ is a Planck parameter and $\mu \gg 1$ is a coupling parameter. I assume that A is a self-adjoint operator and all the conditions are satisfied in the ball $B(0,1)$, $F = F_{12} g^{-\frac{1}{2}}$, $F_{12} = \partial_{x_1} V_2 - \partial_{x_2} V_1$, $g = \det(g^{jk})^{-1}$.

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Further, exactly as in [Ivr10], I assume that

$$F \asymp |x_1|^{\nu-1}, \qquad \nu \in \mathbb{Z}^+,\ \nu \ge 2 \tag{6.2}$$

and thus with no loss of the generality I can assume that

$$V_1 = 0, \qquad V_2 \asymp x_1^{\nu}. \tag{6.3}$$

Furthermore, I assume that either

$$\pm W \ge \epsilon_0, \qquad\qquad \text{as } l \ge 0 \tag{6.4$_\pm$}$$

and as $l = 0$ only sign "+" is interesting or

$$|\partial_{x_2} W/f| \ge \epsilon_0, \qquad f \overset{\text{def}}{=} F x_1^{1-\nu}. \tag{6.5}$$

Also as in [Ivr10], I am interested in the asymptotics of $\int e(x,x,0)\psi(x)\,dx$ where $e(x,y,\tau)$ is the Schwartz kernel of the spectral projector $E(\tau)$ of A and $\psi \in C_0^\infty(B(0,\frac{1}{2}))$ is a cut-off function and I expect the main part of it to be $\int \mathcal{E}^{\mathrm{MW}}(x,0)\psi(x)\,dx$ where $\mathcal{E}^{\mathrm{MW}}$ is defined by (0.8) [1)] which is of magnitude μh^{-1}. *I am assuming without mention that ψ is supported in the small but fixed vicinity of $\{x_1 = 0\}$.*

In the sharp contrast to the analysis of Part I the case $\mu \ge Ch^{-\nu}$ is not "forbidden" anymore as well as zone $\mathcal{Z}'' = \{|x_1| \ge \bar{\gamma}_1 \overset{\text{def}}{=} C(\mu h)^{-1/(\nu-1)}\}$. On the contrary, as $\mu h \ge C$ this zone becomes the main contributor to the principal part of asymptotics which now is of magnitude μh^{-1} instead of $h^{-2}(\mu h)^{-1/(\nu-1)}$ as it was in [Ivr10]. Actually I will time to time slightly change the definition of $\bar{\gamma}_1$, replacing it by $\bar{\gamma}_1 = \epsilon(\mu h)^{-1/(\nu-1)}$ and back and changing respectively definition of zones.

Section 6 is devoted to the case of $\mu \le \epsilon h^{-\nu}$. Analysis in zone $\mathcal{Z}' \overset{\text{def}}{=} \{|x_1| \le 2\bar{\gamma}_1\}$ remains basically the same and the main attention is paid here to the formally forbidden zone $\mathcal{Z}''$. The main results here are theorems 6.10, 6.11 and 6.17.

As $\mu \ge \epsilon h^{-\nu}$ this separation to zones is no more reasonable and will be modified. In section 7 I analyze the case of $\epsilon h^{-\nu} \le \mu \le Ch^{-\nu}$. The main results here are theorems 7.3 and 7.4.

Further, in section 8 analyze the case of $\mu \ge Ch^{-\nu}$. The main results here are theorems 8.9, 8.10, 8.11 and 8.12.

Finally, appendix A is devoted to asymptotics of some one-dimensional Schrödiunger operators associated with (6.1).

[1)] References by default are to [Ivr10].

6.2 Simple Rescaling

As in [Ivr10] the simple rescaling arguments help us to get the easy but not sharp results.

6.2.1 In this and the next subsubsection I assume that $\mu \le Ch^{-\nu}$.Rescaling arguments in the zone $\mathcal{Z}'$ work exactly in the same manner as in [Ivr10] leading to the asymptotics of $\int e(x,x,0)\psi'(x)\,dx$ with the principal part $\int \mathcal{E}^{\mathrm{MW}}(x,0)\psi'(x)\,dx$ and the remainder estimate $O(h^{-1})$ where $\psi'(x)$ and $\psi''(x)$ are cut-off functions supported in zones $\mathcal{Z}'$ and $\mathcal{Z}''$ (defined as above) respectively; one can take $\psi'(x)=\psi(x)\psi_0(x_1/\bar{\gamma}_1)$, $\psi''=\psi-\psi'$ where $\psi_0\in C_0^\infty$ is supported in $(-1,1)$ and equals 1 in $[-\frac{1}{2},\frac{1}{2}]$.

However the contribution of the previously forbidden zone $\mathcal{Z}''$ to the remainder estimate is

$$O\Bigl(\int_{\{\bar{\gamma}_1\le\gamma\le 1\}}\gamma^{-2}\,d\gamma\Bigr)=O\bigl(\bar{\gamma}_1^{-1}\bigr)$$

which is $O\bigl(h^{-1}\bigr)$ due to assumption $\mu\le Ch^{-\nu}$ and the contribution of $\mathcal{Z}''$ to the principal part is

$$\int \mathcal{E}^{\mathrm{MW}}(x)\psi''(x)\,dx=\frac{1}{4\pi}\mu h^{-1}l_\pm\int \psi''|F|\sqrt{g}\,dx,\qquad l_\pm\stackrel{\mathrm{def}}{=}l+\frac{1}{2}(-1\pm 1)\tag{6.6}$$

under condition $(6.4)_\pm$.

Under condition (6.5) the above arguments remain true for the contribution of the subzone $\mathcal{Z}''\cap\{|W|\ge C\gamma\}$; for the contribution of the zone $\mathcal{Z}''\cap\{|W|\le C\gamma\}$ one needs to take in account correction term[2)] $\sum_m \kappa_m\mu_{\mathrm{eff}}h_{\mathrm{eff}}^{1+2m}$ for the case $\mu_{\mathrm{eff}}h_{\mathrm{eff}}\ge 1$, $h_{\mathrm{eff}}\le 1$ where in the rescaling and division arguments $\mu_{\mathrm{eff}}=\mu\gamma^{\nu-\frac{1}{2}}$, $h_{\mathrm{eff}}=h\gamma^{-\frac{3}{2}}$ and the number of balls is $O(1)$ for each γ. Then the total contribution of this correction terms is $O\bigl(\mu h\bigr)$ as $\nu\ge 3$ and $O\bigl(\mu h|\log h|\bigr)$ as $\nu=2$.

6.2.2 Replacing ψ by $x_1\psi$ in the above arguments one gains factor γ in each integrand; then the total contribution of the zone $\mathcal{Z}'$ to the remainder estimate becomes

$$O\Bigl(\int \mu^{-1}h^{-1}\gamma^{1-\nu}\times\gamma\times\gamma^{-2}\,d\gamma\Bigr)=O\bigl(\mu^{-1/\nu}h^{-1}\bigr)$$

which is exactly what I want. On the other hand, the contribution of zone $\mathcal{Z}''$ to the remainder estimate becomes $O(\gamma^{-1}\,d\gamma)=O(|\log h|)$ which is what we want as $\mu\le C(h|\log h|)^{-\nu}$ only. To fix it under condition $(6.4)_\pm$ one can notice that zone $\mathcal{Z}''$ is the spectral gap and therefore the contribution of the individual ball to the remainder estimate is $O(\gamma h_{\mathrm{eff}}^s)$ with $h_{\mathrm{eff}}=h/\gamma$ rather than $O(1)$ and therefore the total contribution of zone $\mathcal{Z}''$ to the remainder estimate is $O(1)$.

[2)] See section 6 of [Ivr1].

As before, under condition (6.5) these arguments are applicable in the subzone $\mathcal{Z}'' \cap \{|W| \ge C\gamma\}$ with $h_{\text{eff}} = h/(\gamma |\mathrm{W}|^{1/2})$ as long as $h_{\text{eff}} \le 1$. This leads to $O(1)$ estimate of the contribution of the subzone $\mathcal{Z}'' \cap \{|W| \ge C\gamma, |W|^{1/2}\gamma \ge h\}$ to the remainder. One can see easily that the integral of γ^{-1} taken over subzones $\mathcal{Z}'' \cap \{|W| \ge C\gamma, |W|^{1/2}\gamma \le h\}$ and $\mathcal{Z}'' \cap \{|W| \le C\gamma\}$ is $O(1)$ as well. Thus rescaling arguments provide remainder estimate $O\bigl(\mu^{-1/\nu} + 1\bigr)$ if ψ contains an extra factor x_1 and under condition (6.5) correction terms are taken into account.

Therefore

(6.7) As $\mu \le Ch^{-\nu}$ in what follows one can assume without any loss of the generality that $\psi(x) = \psi_1(x_1)\psi_2(x_2)$.

6.2.3 As $\mu \ge Ch^{\nu}$ arguments of subsubsection 6.2.1 work as $\{|x_1| \ge Ch\}$ providing $O(h^{-1})$ contribution of this zone to the remainder estimate while the contribution of zone $\{|x_1| \le Ch\}$ will be $O(\mu h^{\nu-1})$. The main part of the asymptotics will be the same as above.

Moreover, arguments of subsubsection 6.2.1 work as $\{|x_1| \ge Ch\}$ providing $O(1)$ contribution of this zone to the remainder estimate as ψ is replaced by $x_1\psi$ while the contribution of zone $\{|x_1| \le Ch\}$ will be $O(\mu h^{\nu})$.

In the next section I will improve these latter results.

6.3 Estimates. I

In section 2 and subsections 4.1–4.4 of [Ivr10] various properties of operator A were proven in the outer and inner zones $\mathcal{Z}_{\text{out}} = \{\bar\gamma \le |x_1| \le 2\bar\gamma_1\}$ and $\mathcal{Z}_{\text{inn}} = \{|x_1| \le 2\bar\gamma\}$ with $\bar\gamma \stackrel{\text{def}}{=} C\mu^{-1/\nu}$ as long as $\bar\gamma \le \bar\gamma_1$ i.e. $\mu \le \epsilon h^{-\nu}$. These properties were proven first in section 2 under assumption

$$C \le \mu \le \epsilon (h|\log h|)^{-\nu} \tag{6.8}$$

using standard microlocal analysis with logarithmic uncertainty principle and then in subsections 4.1–4.4 under assumption

$$\epsilon (h|\log h|)^{-\nu} \le \mu \le \epsilon h^{-\nu} \tag{6.9}$$

applying microlocal analysis for h-pseudo-differential operators with respect to x_2 with operator-valued symbols – operators in the auxiliary space $\mathbb{H} = L^2(\mathbb{R}_{x_1})$; I remind that in te case (6.9) localization was done with respect to ξ_2 rather x_1.

Therefore in both cases (6.8), (6.9) in the redefined outer zone

$$\mathcal{Z}_{\mathrm{out}} = \bigl\{\bar{\gamma} \le |x_1| \le \bar{\gamma}'_1 = \epsilon\bar{\gamma}_1\bigr\} \tag{6.10}$$

(with the small constant ϵ) all these arguments remain true leading us eventually to the following statements:

Proposition 6.1. *Let conditions* (6.2) *and* $(6.4)_+$ *be fulfilled. Let* $\psi = \psi(x_2)$ *be supported in* $B(0, \frac{1}{2})$ *and let* $\varphi = \varphi(\xi_2)$ *be supported in the strip*

$$\mathcal{Y}_\gamma = \bigl\{\mu\gamma^\nu \le |\xi_2| \le 2\mu\gamma^\nu\bigr\} \tag{6.11}$$

with $C_1\bar{\gamma} \le \gamma \le \epsilon_1\bar{\gamma}_1$ *Then*
(i) As $\mu \le \epsilon h^{-\nu}$ *estimates*

$$|F_{t\to h^{-1}\tau}\chi_T(t)\Gamma(Qe)| \le Ch^s \tag{6.12}$$

and

$$\mathcal{R}' = |\Gamma\bigl(Qe\bigr) - h^{-1}\int_{-\infty}^{0} \Bigl(F_{t\to h^{-1}\tau}\bar{\chi}_T(t)\Gamma(Qe)\Bigr)\,d\tau| \le C\mu^{-1}\gamma^{1-\nu}h^{-1} \tag{6.13}$$

hold with $Qe = \varphi(hD_2)\bigl(e\psi\bigr)$, $e = e(x,y,\tau)$, $|\tau| \le \epsilon$, $T \in [T_0, T_1]$, $T_0 = Ch|\log h|$, $T_1 = \epsilon\mu^{-1}\gamma^{-\nu}$;
(ii) Moreover, under condition (6.8) *statement (i) holds with* $Q = \psi_1\psi$, $\psi_1 = \psi_1(x_1)$ *supported in* $\mathcal{Z}_\gamma = \{\gamma \le |x_1| \le 2\gamma\}$.

Corollary 6.2. *Let conditions* (6.2) *and* $(6.4)_+$ *be fulfilled. Let* $\psi = \psi(x_2)$ *be supported in* $B(0, \frac{1}{2})$ *and* $\varphi = \varphi(\xi_2)$ *be supported in the outer zone defined in the terms of* ξ_2

$$\mathcal{Y}_{\mathrm{out}} = \bigl\{C_0 \le |\xi_2| \le \epsilon\bigl(\mu h^\nu\bigr)^{-1/(\nu-1)}\bigr\}. \tag{6.14}$$

Then
(i) As $\mu \le \epsilon h^{-\nu}$ *estimate*

$$\mathcal{R}' \le C\mu^{-1/\nu}h^{-1} \tag{6.15}$$

holds.
(ii) Moreover, under condition (6.8) *statement (i) holds with* $Q = \psi_1\psi$, $\psi_1 = \psi_1(x_1)$ *supported in* $\mathcal{Z}_{\mathrm{out}}$.

On the other hand, under condition $(6.4)_-$ the whole zone $\mathcal{Z}' = \mathcal{Z}_{\mathrm{inn}} \cup \mathcal{Z}_{\mathrm{out}}$ will be forbidden leading us to the following statement not having analogues in [Ivr10]:

Proposition 6.3. *Let conditions* (6.2) *and* $(6.4)_-$ *be fulfilled. Let* $\psi = \psi(x)$, $\psi_1 = \psi_1(x_1)$ *be supported in* $B(0, \frac{1}{2})$ *and* $\mathcal{Z}'$ *respectively and let* $\varphi = \varphi(\xi_2)$ *be supported in the zone*

$$\mathcal{Y}' = \{|\xi_2| \le \epsilon \bigl(\mu h^\nu\bigr)^{-1/(\nu-1)}\}. \tag{6.16}$$

Then
(i) $|Qe| \le Ch^s$ *with* $Qe = \varphi(hD_2)\bigl(e\psi\bigr)$, $e = e(x,y,\tau)$, $|\tau| \le \epsilon$ *as* $\mu \le \epsilon h^{-\nu}$*;*
(ii) Moreover, under condition (6.8) *statement (i) holds with* $Q = \psi_1\psi$, $\psi_1 = \psi_1(x_1)$ *supported in* $\mathcal{Z}'$*.*

Therefore as $\mu \le \epsilon h^{-\nu}$ and condition $(6.4)_+$ is fulfilled one needs to discuss the contribution of the inner zone $\mathcal{Z}_{\mathrm{inn}} = \{|x_1| \le \bar{\gamma}\}$ or equivalently $\mathcal{Y}_{\mathrm{inn}} = \{|\xi_2| \le C_0\}$ [3)] to the remainder estimate. Furthermore one needs to consider the contribution of the previously forbidden zone $\mathcal{Z}'' = \{|x_1| \ge \bar{\gamma}'_1\}$ or equivalently $\mathcal{Y}'' = \{|\xi_2| \ge \epsilon\bigl(\mu h^\nu\bigr)^{-1/(\nu-1)}\}$ [3)] to the remainder estimate.

The inner zone is analyzed exactly as in section 2 and subsections 4.1–4.4 of [Ivr10] leading us eventually to

Proposition 6.4. *Let conditions* (6.2) *and* $(6.4)_+$ *be fulfilled. Let* $\psi = \psi(x_2)$ *and* $\psi_1 = \psi_1(x_1)$ *be supported in* $B(0, \frac{1}{2})$ *and* $\mathcal{Z}_{\mathrm{inn}}$ *respectively and let* $\varphi = \varphi(\xi_2)$ *be supported in* $\mathcal{Y}_{\mathrm{inn}} = \{|\xi_2| \le C_0\}$*. Then all the results of section 2 and subsections 4.1–4.4 of* [Ivr10] *remain true; in particular*
(i) As $\mu \le Ch^{\delta-\nu}$

$$\mathcal{R}'' \overset{\mathrm{def}}{=} |\Gamma\bigl(Qe\bigr) - h^{-1}\sum_j \int_{-\infty}^0 \Bigl(F_{t\to h^{-1}\tau}\bar{\chi}_{T_j}(t)\Gamma(Q_je)\Bigr)\,d\tau| \le C\mu^{-1/\nu}h^{-1} \tag{6.17}$$

with $Qe = \varphi(hD_2)\bigl(e\psi\bigr)$, $e = e(x,y,\tau)$, $Q = \sum_j Q_j$ *and* $|\tau| \le \epsilon$ *where partition* Q_j *and* $Ch|\log h| \le T_j$ *are defined following formula* (3.28) *in* [Ivr10]*;*
(ii) Moreover, under nondegeneracy condition

$$\sum_{1\le k\le m} |\partial_{x_2}^k \bigl(\frac{W}{f}\bigr)| \ge \epsilon_0. \tag*{$(6.18)_m$}$$

[3)] These two definitions are essentially equivalent under condition (6.8) but in the case (6.9) one needs always use definition in the frames of ξ_2.

$\mathcal{R}''$ does not exceed $C\mu^{-1/\nu}h^{-1}$ as $\mu \le \epsilon h^{-\nu}$;

(iii) On the other hand, in the general case $\mathcal{R}''$ does not exceed $C\mu^{-1/\nu}h^{-1} + Ch^{-\delta}$ as $\mu \le \epsilon h^{-\nu}$;

(iv) Furthermore, under condition (6.8) all statements (i)–(iii) hold with $Q = \psi_1\psi$.

Remark 6.5. In frames of proposition 6.4 estimate (6.12) holds for $Q = Q_m$ and $T \in [T_m, T'_m]$ with T'_m defined by (2.98) (it was denoted by T_1 then).

6.4 Estimates. II

To investigate zone $\mathcal{Z}''$ I will apply the theory of operators with operator-valued symbols. However, as $\mu \le \epsilon(h|\log h|)^{-\nu}$ one can apply a usual microlocal analysis with logarithmic uncertainty principle.

So, let us consider A as h-pseudo-differential operator $\mathcal{A}(x_2, hD_2)$ with operator-valued symbol $\mathcal{A}(x_2,\xi_2)$. However, before doing this one can assume without any loss of the generality that $g^{11} = 1$, $g^{12} = 0$ and therefore

$$\mathcal{A}(x_2,\xi_2) = \frac{1}{2}\Bigl(h^2D_1^2 + \sigma^2(x)\bigl(\xi_2 - \mu V_2(x)\bigr)^2 - (2l+1)\mu hF - W(x)\Bigr), \qquad V_2 = \phi(x)\frac{1}{\nu}x_1^\nu \tag{6.19}$$

with $\phi(x) = 1$ as $x_1 = 0$; then $f = \sigma\phi$.

Further, for given x_2 by change of variable x_1 one can transform $\mathcal{A}$ unitarily to the similar operator with $\phi = 1$ and with

$$\sigma = 1 \qquad \text{as } x_1 = 0; \tag{6.20}$$

but this new operator is multiplied from the left and the right by $\alpha(x)$. So operator $\mathcal{A}(x_2,\xi_2)$ is unitary equivalent to

$$\mathcal{A}'(x_2,\xi_2) = \frac{1}{2}\alpha(x)\Bigl(h^2D_1^2 + \sigma^2(x)\Bigl(\xi_2 - \mu\frac{1}{\nu}x_1^\nu\Bigr)^2 - (2l+1)\mu h\sigma(x)x_1^{\nu-1} - W_0(x)\Bigr)\alpha(x). \tag{6.21}$$

Note that $W_0 = W/f$ as $x_1 = 0$ and thus *conditions* $(6.4)_\pm$, *(6.5) and* $(6.18)_m$ *are reformulated in terms of* W_0 obviously.

Proposition A.3(ii) of Appendix A implies that under condition $(6.4)_\pm$ zone $\mathcal{Y}'' \setminus \mathcal{Y}''_0 = \{\epsilon(\mu h^\nu)^{-1/(\nu-1)} \le |\xi_2| \le 2C(\mu h^\nu)^{-1/(\nu-1)}\}$ is microhyperbolic with respect to ξ_2 and thus one can extend $\mathcal{Y}'$ to zone $\bar{\mathcal{Y}}' \overset{\mathrm{def}}{=} \{|\xi_2| \le 2C(\mu h^\nu)^{-1/(\nu-1)}\}$ resulting in the following statement:

Proposition 6.6. *Let conditions (6.2) and*$(6.4)_\pm$ *be fulfilled. Then estimate $\mathcal{R}' \le C$ holds as $\mathcal{R}'$ is defined by (6.13) with $Qe = \varphi(hD_2)(e\psi)$, φ supported in the zone $\mathcal{Y}'' \setminus \mathcal{Y}''_0$, $T \in [T_0, T_1]$, $T_0 = Ch|\log h|$, $T_1 = \epsilon(\mu h^\nu)^{-1/(\nu-1)}$, $\mu \le \epsilon h^{-\nu}$.*

Furthermore, proposition A.3(i) implies that under condition $(6.4)_\pm$ zone $\mathcal{Y}''_0 = \{|\xi_2| \ge C(\mu h)^{-1/(\nu-1)}\}$ is forbidden on energy levels $|\tau| \le \epsilon$ as long as $\mu \le \epsilon h^{-\nu}$ is forbidden; namely

$$|F_{t\to h^{-1}\tau}\bar\chi_T(t)(Qu)(x,y,t)| \le CTh^s \qquad \forall \tau : |\tau| \le \epsilon \tag{6.22}$$

as $Q\psi = \varphi(hD_2)(u\psi)$ with φ supported in the zone $\mathcal{Y}''_0$ and therefore its contribution to the remainder $\mathcal{R}'$ defined by (6.13) is negligible as well:

Proposition 6.7. *Let conditions (6.2) and* $(6.4)_\pm$ *be fulfilled. Then estimate $\mathcal{R}' \le Ch^s$ holds as $\mathcal{R}'$ is defined by (6.13) with $Qe = \varphi(hD_2)(e\psi)$, φ supported in the zone $\mathcal{Y}''_0$, $T \ge T_0 = Ch|\log h|$, $\mu \le \epsilon h^{-\nu}$.*

The analysis of all zones under condition (6.5) will be done in subsection 6.7.

6.5 Calculations. I

In this subsection I will change partition: instead of $\mathcal{Z}'$ and $\mathcal{Z}''$ I will consider $\bar{\mathcal{Y}}'$ and $\mathcal{Y}''_0$ obtained if I redefine $\bar\gamma_1 = C(\mu h)^{-1/(\nu-1)}$; respectively change definitions and notations of zones $\mathcal{Y}_{\rm out}$, $\mathcal{Z}_{\rm out}$, $\mathcal{Z}'$, $\mathcal{Z}''$.

After estimates were derived in two previous subsections under assumption $Ch^{-1} \le \mu \le \epsilon h^{-\nu}$ and condition $(6.4)_\pm$ calculations in zone $\bar{\mathcal{Y}}'$ are done exactly as in section 3 and subsection 4.4 of [Ivr10].

On the other hand, calculations in zone $\mathcal{Y}''_0$ as $\mu \le \epsilon h^{-\nu}$ are rather obvious under assumptions $Ch^{-1} \le \mu \le \epsilon h^{-\nu}$ and $(6.4)_\pm$. Therefore I arrive to the intermediate estimate

$$|\int \Bigl(\bigl(\varphi(hD_2)e\bigr)(x,x,0) \ -(2\pi h)^{-1}\int \mathrm{e}(x_1,x_1;x_2,\xi_2,0)\varphi(\xi_2)\,d\xi_2\Bigr)\psi_2(x_2)\,dx| \le R \tag{6.23}$$

where R is an estimate already derived in the corresponding conditions (also see below) and $\varphi \in C_0^\infty(-\epsilon', \epsilon')$ with sufficiently small constant ϵ'.

Then the same estimate holds with $\psi(x_2)$ replaced by $\psi(x)$ such that $\psi(x) = \psi_2(x_2)$ as $|x_1| \le C_1\epsilon'$ because this transition leads to a negligible error. I take ψ also satisfying $\psi(x) = 0$ as $|x_1| \ge 2C_1\epsilon'$. Then in the latter estimate I can replace φ by 1. Really, then the error would be

$$|\int \Bigl(\bigl((1-\varphi(hD_2))e\bigr)(x,x,0) \ -(2\pi h)^{-1}\int \mathrm{e}(x_1,x_1;x_2,\xi_2,0)(1-\varphi(\xi_2))\,d\xi_2\Bigr)\psi(x)\,dx| \tag{6.24}$$

and replacing ψ by ψ' equal to ψ as $|x_1| \ge 2C_2^{-1}\epsilon'$ and equal to 0 as $|x_1| \le C_2^{-1}\epsilon'$ leads to a negligible error. However, to expression (6.24) modified this way one can apply the theory of operators with non-degenerating magnetic field and then to estimate expression (6.24) by C.

Thus I derived (6.23) with φ replaced by 1 and $\psi_2(x_2)$ replaced by some "special" function $\psi(x)$. Then due to rescaling arguments like in subsubsection 6.2.2 the same estimate holds for a general function $\psi(x)$ supported in $\{|x_1| \le 2C_1\epsilon'\}$. Thus I arrive to

Proposition 6.8. *Let conditions (6.2) and $(6.4)_+$ be fulfilled. Then*
(i) As <u>*either*</u> $\mu \le h^{\delta-\nu}$ <u>*or*</u> *condition* $(6.18)_m$ *is fulfilled and* $\mu \le \epsilon h^{-\nu}$ *the following estimate holds*

$$\mathcal{R}_I \stackrel{\mathrm{def}}{=} |\int \Bigl(\mathrm{e}(x,x,0) \ - (2\pi h)^{-1} \int \mathrm{e}(x_1,x_1;x_2,\xi_2,0)\,d\xi_2\Bigr)\psi(x)\,dx| \le C\mu^{-\frac{1}{\nu}}h^{-1} \tag{6.25}$$

where here and below $\mathrm{e}(x_1,y_1;x_2,\xi_2,\tau)$ *is the Schwartz kernel of the spectral projector of operator* $\mathcal{A}(x_2,\xi_2)$ *defined by (6.19) and* $\delta > 0$ *is an arbitrarily small exponent;*
(ii) In the general case with $\mu \le \epsilon h^{-\nu}$ *estimate*

$$\mathcal{R}_I \le C\mu^{-\frac{1}{\nu}}h^{-1} + Ch^{-\delta} \tag{6.26}$$

holds.

I remind that in both statements of proposition 6.8 the principal part of asymptotics has magnitude $\asymp \mu h^{-1}$ (as $\mu \ge h^{-1}$).

On the other hand, under condition $(6.4)_-$ zone $\mathcal{Y}'$ becomes forbidden and thus I arrive to

Proposition 6.9. *Let conditions (6.2) and $(6.4)_-$ be fulfilled and $l \ge 1$. Then for $Ch^{-1} \le \mu \le \epsilon h^{-\nu}$ estimate $\mathcal{R}_I \le C$ holds while the principal part of asymptotics has magnitude $\asymp \mu h^{-1}$.*

6.6 Calculations. II

Transition to the auxiliary operator $\mathcal{A}_0$ without increasing error estimates could be done easily in zone $\mathcal{Y}_{\mathrm{out}}$ exactly as it was done in the proof of propositions 3.3 and 3.4 while arguments of 3.8 etc work in zone $\mathcal{Y}_{\mathrm{inn}}$.

On the other hand, this transition in zone $\mathcal{Y}_0''$ is obvious under condition $(6.4)_\pm$, and I arrive to two theorems below as $\mu \le h^{-\nu}|\log h|^{-K}$ and function ψ is "special" in the sense

of the previous subsection. Then the same arguments as there extend theorem to general ψ.

Furthermore, under condition $(6.4)_\pm$ the case $h^{-\nu}|\log h|^{-K} \le \mu \le \epsilon h^{-\nu}$ is analyzed exactly as in section 4 of Part I leading to the extension of these theorems to $\mu \le \epsilon h^{-\nu}$:

Theorem 6.10. *Let conditions (6.2) and $(6.4)_+$ be fulfilled. Then*
(i) As <u>*either*</u> $\mu \le h^{\delta-\nu}$ <u>*or*</u> *condition* $(6.18)_m$ *is fulfilled and* $\mu \le \epsilon h^{-\nu}$

$$\mathcal{R}^* \overset{\mathrm{def}}{=} |\int \Bigl(e(x,x,0) - \tilde{\mathcal{E}}^{\mathrm{MW}}(x,0)\Bigr)\psi(x)\,dx - \int \mathcal{E}^{\mathrm{MW}}_{\mathrm{corr}}(x_2,0)\psi(0,x_2)\,dx_2| \tag{6.27}$$

does not exceed $C\mu^{-1/\nu}h^{-1}$ *where*

$$\mathcal{E}^{\mathrm{MW}}_{\mathrm{corr}}(x,\tau) \overset{\mathrm{def}}{=} (2\pi h)^{-1}\int \mathrm{e}_0(x_1,x_1;x_2,\xi_2,\tau,\hbar)\,dx_1 d\xi_2 - \int \tilde{\mathcal{E}}^{\mathrm{MW}}_0(x,\tau)\,dx_1, \tag{6.28}$$

$\mathcal{E}^{\mathrm{MW}}$ *is Magnetic Weyl approximation*[4] *and here and below* $\mathrm{e}_0(x_1,y_1;x_2,\xi_2,\tau)$ *is the Schwartz kernel of the spectral projector of operator* $\mathcal{A}_0(x_2,\xi_2)$ *defined by (6.19) and with* α,ϕ,σ,W *restricted to* $\{x_1=0\}$ *and* $\mathcal{E}^{\mathrm{MW}}_0$ *is Magnetic Weyl approximation for this operator.*
(ii) In the general case with $\mu \le \epsilon h^{-\nu}$ *estimate* $\mathcal{R}^* \le C\mu^{-1/\nu} + Ch^{-\delta}$ *holds.*

Theorem 6.11. *Let conditions (6.2) and $(6.4)_-$ be fulfilled and $l \ge 1$. Then as $Ch^{-1} \le \mu \le \epsilon h^{-\nu}$ estimate $\mathcal{R}^* \le C$ holds while the principal part of asymptotics has magnitude $\asymp \mu h^{-1}$.*

Remark 6.12. Obviously the same approximate expressions (3.52), $(3.52)^*$, $(3.52)^{**}$ hold for the part of $\mathcal{E}^{\mathrm{MW}}_{\mathrm{corr}}$ "associated" with $\mathcal{Y}_{\mathrm{inn}}$;

6.7 Estimates under condition (6.5)

I start from the remainder estimate in zone $\bar{\mathcal{Y}}'$ which is trivial:

Proposition 6.13. *Let conditions (6.2), (6.20) and (6.5) be fulfilled. Then*
(i) Estimate (6.13) holds with $Qe = \varphi(hD_2)(e\psi)$, φ *supported in the strip* $\mathcal{Y}_\gamma$ *with the same restrictions to* γ *and the same* T_0, T_1 *as in proposition 6.1(i);*
(ii) Furthermore, the same estimate holds as φ *is supported in zone* $\mathcal{Y}_{\mathrm{inn}}$ *and* $\gamma = \bar{\gamma}_0 = \mu^{-1/\nu}$*;*
(iii) Therefore $\mathcal{R}'$ *defined by (6.13) does not exceed* $C\mu^{-1/\nu}h^{-1}$ *as* φ *is supported in zone* $\bar{\mathcal{Y}}'$ *and* $T = T_0$.

[4] See e.g. (0.8).

Now let us analyze zone $\mathcal{Y}_0''$ under condition (6.5):

Proposition 6.14. *Let conditions (6.2), (6.20) and (6.5) be fulfilled. Then estimate $\mathcal{R}' \le C$ holds as $\mathcal{R}'$ is defined by (6.13) with $Qe = \varphi(hD_2)(e\psi)$, φ supported in the zone $\mathcal{Y}_0''$.*

Proof. (i) Let us note first that estimate

$$|F_{t\to h^{-1}\tau}\bigl(\bar\chi_{T_1}(t) - \bar\chi_{\bar T}(t)\bigr)(Qu)(x,y,t)| \le Ch^s \qquad \forall \tau : |\tau| \le \epsilon \tag{6.29}$$

holds with $T_1 = \epsilon\mu^{-1}\gamma^{-\nu}$, $\bar T = Ch|\log h|$ as $Qu = \varphi(hD_2)(u\psi)$, φ supported in the strip $\mathcal{Y}_{(\gamma)} = \{\mu\gamma^\nu \le |\xi_2| \le 2\gamma^\nu\}$ with $\gamma \ge C\bar\gamma_1$.

Really, us consider a partial trace $\Gamma'(Qu)$ (with respect to x_1). Due to proposition A.3 the propagation speed with with respect to x_2 does not exceed $C|\xi_2|^{-1} \asymp C(\mu\gamma^\nu)^{-1}$ and the propagation speed with respect to ξ_2 does not exceed C [5)]; moreover, under condition (6.5) this propagation speed with respect to ξ_2 is greater than ϵ.

On the other hand, an obvious estimate

$$|F_{t\to h^{-1}\tau}\bar\chi_{T_0}(t)\Gamma(Qu)(t)| \le C\mu\gamma^\nu h^{-1} \times T_0 = C\mu\gamma^\nu|\log h| \tag{6.30}$$

holds where the first factor is $\mu_{\mathrm{eff}} h_{\mathrm{eff}}^{-1}\gamma^{-1}$; furthermore, due to (6.29) this estimate holds for the left-hand expression with T_0 replaced by T_1.

Therefore the contribution of the strip $\mathcal{Y}_\gamma$ to the remainder estimate does not exceed

$$C\mu\gamma^\nu|\log h| \times T_1^{-1} = C|\log h| \tag{6.31}$$

and therefore the total contribution of $\mathcal{Y}_0''$ to the remainder estimate does not exceed $C|\log h|\int \gamma^{-1}\,d\gamma \asymp C|\log h|^2$.

This estimate is as good as I need for $\mu \le Ch^{-\nu}|\log h|^{-2\nu}$. However for $Ch^{-\nu}|\log h|^{-2\nu} \le \mu \le \epsilon h^{-\nu}$ I would like to improve it getting rid of two logarithmic factors.

(ii) Getting rid off one of them is easy: rescaling $t \mapsto t/T$, $(x_j - y_j) \mapsto (x_j - y_j)/T$, $\mu \mapsto \mu T$, $h \mapsto h/T$ estimates for Schrödinger operator with strong non-degenerate magnetic field [Ivr1], section 6 (with arbitrary parameters μ and h such that $\mu h \ge C$) I arrive to two following inequalities

$$|F_{t\to h^{-1}\tau}\chi_T(t)\Gamma(Qu)| \le C\mu\bigl(\frac{h}{T}\bigr)^s \tag{6.32}$$

$$|F_{t\to h^{-1}\tau}\bar\chi_T(t)\Gamma(Qu)| \le C\mu \tag{6.33}$$

[5)] Under some assumptions this would be equivalent to the estimate of the the average propagation speed with respect to x_1 of Qu by $C\gamma(\mu\gamma^\nu)^{-1}$; further one can estimate average propagation speed with respect to x_2 of Qu by $C(\mu\gamma^\nu)^{-1}$ as well.

as $h \le T \le 1$, $|\tau| \le \epsilon$ under condition $|W| + |\nabla W| \ge \epsilon_0$. Then using our standard scaling $x_1 \mapsto x_1/\gamma$, $x_2 \mapsto (x_2 - y_2)/\gamma$, $\mu \mapsto \mu_{\text{eff}} = \mu\gamma^\nu$, $h \mapsto h_{\text{eff}} = h/\gamma$ and $T \mapsto T/\gamma$ I arrive to estimate (6.30) without logarithmic factor

$$|F_{t\to h^{-1}\tau}\bar{\chi}_T(t)\Gamma(Qu)(t)| \le C\mu\gamma^\nu \tag{6.30*}$$

as $|\tau| \le \epsilon$, $T/\gamma \le \epsilon\mu\gamma^\nu \iff T \le T_1' = \epsilon\mu\gamma^{\nu+1}$. Further, and to (6.29) this estimate holds as $h \le T \le T_1 = \epsilon\mu\gamma^\nu$ provided $T_1' \ge Ch$ i.e. $\gamma \ge \bar{\gamma}_1$.

Then the contribution of the strip $\mathcal{Y}_\gamma$ to the remainder $\mathcal{R}'$ is C and therefore the total estimate is $C|\log h|$.

(iii) To get rid off the second logarithmic factor I need to further increase T_1 in the previous arguments and for this purpose I need for each γ to make x_2-partition of $\mathcal{Y}_\gamma$ of the size

$$\ell = \epsilon|V(0,x_2)| + \bar{\ell}, \qquad \bar{\ell} \ge C\gamma. \tag{6.34}$$

Consider first elements $\mathcal{U}_{\gamma,\ell}$ with $\ell \ge C\bar{\ell}$. For every such element on levels τ with $|\tau| \le \epsilon\ell$ after rescaling

$$x_2 \mapsto x_2\ell^{-1}, \quad h \mapsto h' = h\ell^{-\frac{3}{2}}, \quad t \mapsto t\ell^{-1}, \quad \mu \mapsto \mu' = \mu\ell^{\frac{1}{2}} \tag{6.35}$$

I am in the elliptic situation.

Therefore contribution of each such element to the remainder estimate does not exceed $C\mu'(h')^s$ and therefore the total contribution of such elements is negligible as $\bar{\ell} = h^\delta$.

So I need to consider only elements $\mathcal{U}'_\gamma = \mathcal{U}_{\gamma,\ell}$ with $\ell \asymp \bar{\ell} = h^\delta$. For such elements after rescaling (6.35) I can apply estimate (6.30)*; then scaling back I get the same estimate (6.30)* again but with $Q = \psi'(x_2)\varphi(hD_2)$ supported in $\mathcal{U}'_\gamma$, $|\tau| \le \epsilon\ell$ and $Ch|\log h|\ell^{-1} \le T \le T_1 = \epsilon\mu\gamma^{\nu+1}$[6]. Furthermore, applying (6.29) I can increase T_1 to $\epsilon\mu\gamma^\nu$.

So far I gained nothing: the estimate I proved alone would bring me the same final remainder estimate $C|\log h|$ as before but now I can further increase T_1 and thus reduce the remainder estimate.

Namely, let us consider propagation in the time direction in which $|\xi_2|$ increases. If only propagation with respect to ξ_2 was considered, until time $\epsilon_3\mu$ it would be confined to zone

$$\bigl\{\epsilon_0 \le |\xi_2|\bigl(\mu\gamma^\nu + |t|\bigr)^{-1} \le C\bigr\} \subset \bigl\{\frac{1}{2}\mu\gamma^\nu \le |\xi_2| \le \epsilon_1\mu\bigr\}$$

and thus to $\{|x_1| \le \epsilon_3\}$.

[6] It is consistent with the fact that support of ψ' is of the length ℓ but now $\bar{T} = Ch|\log h|/\ell$.

However let us note that the propagation speed with respect to x_2 does not exceed $C\ell/|\xi_2|$ as $\ell \ge C|V| + \bar{\ell}$. Therefore one can prove easily that propagation, which started in the zone $\{|x_2| \le \frac{1}{2}, |V| \le h^{\delta}\}$ as I have assumed, until time $T_1^* = \mu\gamma^\nu h^{-\delta_1}$ is confined to a bit larger zone $\{|x_2| \le \frac{3}{4}, |V| \le h^{\delta/2}\}$ of the same type.

Therefore estimate (6.30)* holds with $Ch^{1-2\delta} \le T \le T_1^*$. Then due to the Tauberian approach contribution of each partition element $\mathcal{U}'_\gamma$ to the remainder estimate does not exceed $C\mu\gamma^\nu T_1^{*-1} = Ch^{\delta_1}$ and the contribution of the whole strip $\mathcal{Y}_\gamma$ does not exceed Ch^{δ_1} as well and of the whole zone $\mathcal{Y}''_0$ does not exceed Ch^{δ_2}.

Clearly, at some moment I increased slightly T_0 but after summation over partition was done I can (using negligibility of the trace on $[Ch|\log h|, h^{1-\delta}]$ time interval on energy levels $|\tau| \le \epsilon$) return to original $\bar{T}$. $\square$

6.8 Calculations under condition (6.5)

Calculations in zone $\bar{\mathcal{Y}}'$ are exactly as in [Ivr10]. However one should be more careful with calculations in zone $\mathcal{Y}''_0$.

Let me remind that according to subsection 6.2 [Ivr1] in the nondegenerate case with $\mu h \ge C$ the operator in question is reduced to one-dimensional $\mu^{-1}h$-pdo $B(x_2, \mu^{-1}hD_2, h^2)$ [7] with the "main symbol" $B(x_2, \xi_2, 0) = W \circ \Psi$ and therefore the contribution of the partition element to the final answer will be given as in subsection 6.6 by magnetic Weyl expression $\int \mathcal{E}^{\mathrm{MW}}(x,0)\psi(x)\,dx$ plus correction terms $\mu h^{1+2m}\int \varkappa_{l,m}(x)\psi(x)\,dx$, $m = 0, 1, \dots$.

After rescaling $\mu \mapsto \mu\gamma^\nu$, $h \mapsto h/\gamma$, $dx \mapsto \gamma^{-2}dx$ these terms are transformed into

$$\mu h^{1+2m} \int \varkappa_{l,m}(x,\gamma)\psi(x)\gamma^{\nu-2m-3}\,dx \tag{6.36}$$

integrated over zone $\{\bar{\gamma}_1 \le \gamma \le \epsilon\}$.

One can see easily that if there was an extra factor γ one would be able to rewrite this expression (6.36) modulo $O(1)$ into the similar expression with integration over $\{\gamma \le \epsilon\}$ as $2m+2 < \nu$ [8] or to simply skip it as $2m+2 > \nu$ or to get a term which is $O(\mu h^\nu |\log h|)$ as $2m+2 = \nu$. To gain this extra factor one needs to consider the difference of expressions $\int e(x,x,0)\psi(x)\,dx$ for two operators with $g^{jk}(x)$, $f(x)$, $V(x)$ coinciding as $x_1 = 0$. As this second operator it is natural to pick up the simplest one i.e.

$$A_0 = \frac{1}{2}\Bigl(h^2D_1^2 + \bigl(hD_2 - \mu x_1^\nu/\nu\bigr)^2 - (2l+1)\mu h x_1^{\nu-1} - W(x_2)\Bigr). \tag{6.37}$$

Therefore I arrive to

---

[7] Where x_2 is not our original x_2.

[8] thus resulting in exactly expression $\kappa_{l,m}\mu h^{1+2m}$ as in non-degenerate case.

Proposition 6.15. *Under condition* (*6.5*) *estimate*

$$
\begin{multlined}
|\int \Bigl(e(x,x,0)-e_0(x,x,0)-\mathcal{E}^{\mathrm{MW}}(x,0)+\mathcal{E}_0^{\mathrm{MW}}(x,0)\Bigr)\psi(x)\,dx-\\
\sum \kappa'_{l,m}\mu h^{1+2m}| \le C\mu^{-1/\nu}h^{-1}
\end{multlined}
\tag{6.38}
$$

holds as $\mu \le h^{-\nu}|\log h|^{-K}$ *where* e_0 *and* $\mathcal{E}_0^{\mathrm{MW}}$ *are defined for operator* A_0.

(6.39) Now in what follows I can consider operator A_0 instead of A.

Then I can apply the standard method of successive approximations with unperturbed operator $\mathcal{A}(y_2, hD_2)$ and plug the results of successive approximations into expression

$$
h^{-1}\int_{-\epsilon}^{0}\Bigl(F_{t\to h^{-1}\tau}\bar{\chi}_T(t)\Gamma(Qu)\Bigr)\,d\tau
\tag{6.40}
$$

which calculates exactly contribution of the "problematic" eigenvalue λ_l of the corresponding one-dimensional Schrödinger operator; I remind that $T=\bar{T}=Ch|\log h|$.

Thus while the main part of asymptotics is estimated by $C\mu h^{-2}\gamma^\nu T = C\mu h^{-1}\gamma^\nu|\log h|$, each next term seemingly acquires factor

$$
Ch^{-1}\bigl(\mu h\gamma^{\nu-1}\bigr)^{1/2}T^2 \asymp Ch\bigl(\mu h\gamma^{\nu-1}\bigr)^{1/2}|\log h|^2;
\tag{6.41}
$$

since the propagation speed with respect to x_2 is estimated by $C_0(\mu h\gamma^{\nu-1})^{1/2}$ such factor could be larger than 1.

In fact however, $C_0(\mu h\gamma^{\nu-1})^{1/2}$ is the estimate for the instant propagation speed only. Using instead the mentioned reduction to a one-dimensional $\mu^{-1}h$-pdo one can find that the propagation speed with respect to x_2 is estimated by $C_0\mu^{-1}$ if magnetic field is non-degenerate and then in the canonical coordinates for time $T=\bar{T}$ the shift of (x'_2,ξ'_2) will be estimated by $C_0\bigl(\mu^{-}h|\log h|\bigr)^{1/2}$ which is the smallest distance allowed by the logarithmic uncertainty principle[9)] and this would persist if one returns back to the original $(x_2,\mu^{-1}\xi_2)$; so one would be able to estimate (x_2-y_2) on the time interval in question by $C_0\bigl(\mu^{-1}h|\log h|\bigr)^{1/2}$.

[9)] Since $\mu^{-1}h$-Fourier Integral Operators are involved later one needs the same distance in each (x,ξ) direction.

In the degenerate case described here one must replace μ, h by $\mu\gamma^\nu$, h/γ respectively and then multiply by γ thus producing final estimate for $|x_2 - y_2|$

$$\varrho \stackrel{\mathrm{def}}{=} C\bigl(\mu^{-1}h\gamma^{1-\nu}|\log h|\bigr)^{1/2} \asymp Ch\bar\gamma_1^{\frac{1}{2}(\nu-1)}\gamma^{-\frac{1}{2}(\nu-1)}|\log h|^{\frac{1}{2}} \tag{6.42}$$

and therefore each next term acquires factor $\varrho|\log h|$. Then m-th term of the final answer is estimated by

$$C\mu h^{-1}\varrho^{m-1}|\log h|^K \asymp C\mu h^{m-2}\gamma^{\nu-\frac{1}{2}(\nu-1)(m-1)}\bar\gamma_1^{\frac{1}{2}(\nu-1)(m-1)}|\log h|^K. \tag{6.43}$$

After integration over $\gamma^{-1}\,d\gamma$ with $\bar\gamma_1 \le \gamma \le \epsilon$ expression (6.43) results in $C\mu h^{m-2}\bar\gamma_1^\nu|\log h|^K$ as $\nu - \frac{1}{2}(\nu-1)(m-1) \le 0$ or in $C(\mu^{-1}h)^{(m-3)/2}|\log h|^K$ otherwise. One can check easily that in either case the answer is $O(|\log h|^K)$ as $m \ge 3$ and only terms with $m = 1, 2$ should be considered more carefully under condition (6.8).

On the other hand, the main term appears as (6.40) with u replaced by $\bar u$ and modulo negligible one can rewrite it with any $T \ge \bar T$, in particular with $T = \infty$ which leads to

$$(2\pi h)^{-1}\int \mathrm{e}_0(x_1, x_1, 0; x_2, \xi_2)\psi(x_1)\varphi(\xi_2)\,dx_2\,d\xi_2 \tag{6.44}$$

where I remind that $\mathrm{e}_0(x_1, y_1, 0; x_2, \xi_2)$ is the Schwartz kernel of the spectral projector of one-dimensional Schrödinger operator $\mathcal{A}_0(x_2, \xi_2)$.

Let us consider terms with $m = 2$ i.e. expression (6.40) with u replaced by $\bar u_1$; similarly to analysis of (i) one can estimate contribution of $O\bigl((x_2 - y_2)^2\bigr)$ terms in the perturbation $\mathcal{A}(x_2, hD_2) - \mathcal{A}(y_2, hD_2)$ by $C|\log h|^K$. Therefore one should consider only $\mathcal{A}(x_2, hD_2) - \mathcal{A}(y_2, hD_2) = (x_2 - y_2)B_1(y_2)$ in which case $\bar u_1$ is defined by (3.23) without the last term since B_1 commutes with $(x_2 - y_2)$:

$$u \mapsto \bar u_1 = -ih\sum_{\varsigma=\pm}\varsigma\bar G^\varsigma B_1\bar G^\varsigma[\bar A, x_2 - y_2]\bar G^\varsigma\delta(t)\delta(x_2 - y_2)\delta(x_1 - y_1). \tag{6.45}$$

One needs to multiply this by $h^{-1}\psi$, integrate with respect to τ and apply Γ to it. Obviously since for odd ν operators $\bar G^\varsigma$ and $[\bar A, x_2 - y_2]$ are even and odd respectively as $x_1 \mapsto -x_1$, $\xi_2 \mapsto -\xi_2$ the answer would be 0 if ψ is even with respect to x_1.

To cover the case of even ν and general ψ let us note that B_1 commutes with $\bar G^\varsigma$ considered as operators in the auxiliary space $L^2(\mathbb{R}^1_{x_1})$. Then if $\bar G^\varsigma$ commuted with ψ, taking trace and integrating with respect to τ would result in

$$\mathrm{const}\cdot\partial_{\xi_2}B_1\sum_{\varsigma=\pm}\varsigma\,\mathrm{Tr}\bigl(\bar G^\varsigma\psi\bigr)$$

which after integration over ξ_2 results in 0.

However $\bar{G}^\varsigma$ does not commute with ψ, so instead of 0 one gets

$$\text{const}\cdot B_1 \sum_{\varsigma=\pm} \operatorname{Tr}\varsigma\Bigl(\bar{G}^\varsigma\bigl(\partial_{\xi_2}\bar{G}^\varsigma\bigr)\Bigl(\bar{G}^\varsigma[\bar{A},\psi]\Bigr)$$

and to this expression one can apply the same type of transformations and calculations as in the proof of proposition 6.15 resulting in the expressin $\sum_m \kappa_{l,m}\mu h^{1+2m}$ where coefficients $\kappa_{l,m}$ are changed as needed.

Therefore combining with the results for zone $\bar{\mathcal{Y}}'$ I arrive to

Proposition 6.16. *For a model operator*

$$|\int\Bigl(e_0(x,x,0)-(2\pi h)^{-1}\int \mathrm{e}_0(x_1,x_1,0;x_2,\xi_2)\,d\xi_2\Bigr)\psi(x)\,dx- \\ \sum \kappa_m\mu h^{1+2m}| \le C\mu^{-1/\nu}h^{-1} \tag{6.46}$$

as $\mu \le Ch^{-\nu}|\log h|^{-K}$*.*

Further, combining this with proposition 6.14 I get as $\mu \le h^{-\nu}|\log h|^{-K}$ estimate (6.47):

Theorem 6.17. *Under condition* (*6.5*) *estimate*

$$|\int\Bigl(e(x,x,0)-(2\pi h)^{-1}\int \mathrm{e}_0(x_1,x_1,0;x_2,\xi_2)\,d\xi_2 \\ -\mathcal{E}^{\mathrm{MW}}(x,0)+\mathcal{E}^{\mathrm{MW}}_0(x,0)\Bigr)\psi(x)\,dx-\sum\kappa_{l,m}\mu h^{1+2m}| \le C\mu^{-1/\nu}h^{-1} \tag{6.47}$$

holds as $\mu \le \epsilon h^{-\nu}$*.*

Proof. To finish the proof of this theorem one needs to cover the case $h^{-\nu}|\log h|^{-K} \le \mu \le \epsilon h^{-\nu}$, getting rid of the term $|\log h|^K$ in the error estimates.

The first problematic error comes from the correction terms in proposition 6.15, namely from the terms of the type $\mu h^{1+2m}\int \varkappa_{l,m}(x_2)\gamma^{\nu-2m-3+k}\,dx$ with $k\ge 1$, $\nu-2m-3+k=-1$ and this error term is $O(1)$ unless $k=1$, $\nu=2m+1$ in which case it it is $\kappa'_l\mu h^\nu|\log h|$. This is possible only for odd ν in which case operator $\mathcal{A}_0$ is even with respect to $x_1\mapsto -x_1$, $\xi_2\mapsto-\xi_2$ but perturbation contains exactly one factor x_1 and therefore it is odd and after integration with respect to x_1, ξ_2 this correction term results in 0 if ψ is even with respect to x_1.

Further, one needs to consider terms corresponding to $m = 3$ in the successive approximations leading to proposition 6.16 and there one can replace $\mathcal{A}_0(x_2, \xi_2) - \mathcal{A}_0(y_2, \xi_2)$ by $B_1(x_2 - y_2)$, and also terms corresponding to $m = 2$ in the same successive approximations and there one can replace $\mathcal{A}_0(x_2, \xi_2) - \mathcal{A}_0(y_2, \xi_2)$ by $B_2(x_2 - y_2)^2$.

To calculate the contribution of such terms one can apply the same approach as in the proof of proposition 6.15 and the contribution of γ-admissible partition element with respect to x_1 will be

$$\sum_m \mu h^{1+2m} \int \varkappa_{l,m,k}(x_2) \psi(x) \gamma^{\nu-2m-3+k}\, dx$$

with $k \ge 0$; however since this expression should be $O(|\log h|^K)$ all the terms but those with $\nu \le 2m+1$, $k \ge 1$ should vanish; further, the total contribution of all remaining terms save those with $\nu = 2m + 1$ and $k = 1$ is $O(1)$, which leaves us with no "bad" terms for even ν and with one "bad" term $\kappa'_l \mu h^\nu \log h$ for odd ν, $m = (\nu - 1)/2$. However, parity considerations with respect to x_1 show that this term should vanish as well. □

Remark 6.18. (i) All the coefficients $\varkappa_{l,*}$ and $\kappa_{l,*}$ vanish for $l = 0$.
(ii) Obviously the same approximate expressions (3.52), (3.52)*, (3.52)** as in [Ivr10] hold for part $\mathcal{E}^{\mathrm{MW}}_{\mathrm{corr}}$ "associated" with $\mathcal{Y}_{\mathrm{inn}}$;

7 Modified V. II. $\epsilon_0 h^{-\nu} \le \mu \le C_0 h^{-\nu}$

Now I will consider the intermediate case

$$\epsilon_0 h^{-\nu} \le \mu \le C_0 h^{-\nu} \tag{7.1}$$

with arbitrarily small constant ϵ_0 and arbitrarily large constant C_0; this case which described the largest possible values in [Ivr10] now is no more than transition to the next section.

7.1 Estimates

Let us denote by $\lambda_n(\xi_2)$ eigenvalues of operator

$$\mathbf{a}^0 = \frac{1}{2}\Bigl(D_1^2 + \bigl(\xi_2 - x_1^\nu/\nu\bigr)^2 - (2l+1)x_1^{\nu-1}\Bigr); \tag{7.2}$$

then $\Lambda_n(x_2, \xi_2) = \lambda_n(\xi_2) - \frac{1}{2}W(x_2)$ are eigenvalues of $\mathbf{a} = \mathbf{a}^0 - W(x_2)$.

My main nondegeneracy assumption will be

$$|\Lambda_n| + (|\xi_2|+1)|\partial_{\xi_2}\Lambda_n| + |\partial_{x_2}\Lambda_n| \ge \epsilon_0 \qquad \forall n, \xi_2, \tag{7.3}$$

may be coupled with $(6.4)_\pm$. This condition (7.3) follows from (6.5); further, it follows from $(6.4)_\pm$ for $|\xi_2| \ge C$. On the other hand, since $\lambda_n \to 0$ and $\xi_2\partial_{\xi_2}\lambda_n \to 0$ as $|\xi_2| \to \infty$, condition (7.3) implies that $|W| + |\partial_{x_2} W| \ge \epsilon_0$ and therefore locally one of conditions $(6.4)_\pm$, (6.5) must be fulfilled.

Obviously, under conditions (7.1),(7.3) for each ξ_2 number of eigenvalues of one-dimensional operator

$$\mathcal{A}_0 = \frac{1}{2}\Bigl(h^2 D_1^2 + \bigl(\xi_2 - \mu x_1^\nu/\nu\bigr)^2 - (2l+1)x_1^{\nu-1} - W\Bigr) \tag{7.4}$$

below level c_0 does not exceed C.

Further, note that condition (7.3) for eigenvalues of $\mathcal{A}_0$ is equivalent to the same condition for eigenvalues of $\mathbf{a}$. Then I easily arrive to

Proposition 7.1. *Under conditions (7.1), (7.3) contribution to the remainder estimate of the zone $\{|\xi_2| \le C\}$ is $O(1)$.*

Furthermore, analysis in the zone $\mathcal{Y}_0''$ under condition (7.1) does not differ from the analysis as $\mu \le \epsilon h^{-\nu}$. Namely

(7.5) Under conditions (7.1) and $(6.4)_\pm$ operator $\mathcal{A}_0$ and thus operator $\mathcal{A}$ is elliptic in the zone $\mathcal{Y}_0'' \overset{\mathrm{def}}{=} \{|\xi_2| \ge C\}$ and the contribution of $\mathcal{Y}_0''$ to the remainder estimate is negligible.

(7.6) Similarly, under conditions (7.1) and (6.5) operator $\mathcal{A}$ is microhyperbolic in the zone $\mathcal{Y}_0'' \overset{\mathrm{def}}{=} \{|\xi_2| \ge C\}$ and the contribution of $\mathcal{Y}_0''$ to the remainder estimate is $O(1)$.

Therefore

Proposition 7.2. *Let conditions (7.1), (7.3) and one of conditions $(6.4)_\pm$, (6.5) be fulfilled. Then the remainder estimate is $O(1)$ where the principal part is defined by (6.40).*

7.2 Calculations

Calculations in this case also do not differ from those in section 6 leading to the following statements

Theorem 7.3. *Let conditions (7.1), (7.3) and* $(6.4)_\pm$ *be fulfilled. Then* $\mathcal{R}_I$ *defined by (6.25) and* $\mathcal{R}^*$ *defined by (6.27) do not exceed* C.

Theorem 7.4. *Let conditions (7.1) and (6.5) be fulfilled. Then left-hand expressions of (6.38), (6.46) and (6.47) do not exceed* C.

8 Modified V. III. $\mu \geq C_0 h^{-\nu}$

Now I consider the previously forbidden case

$$\mu \geq C_0 h^{-\nu} \tag{8.1}$$

with sufficiently large constant C_0. In this case all zones should be redefined. Also the difference between $l = 0$ and $l \geq 1$ becomes crucial.

8.1 Estimates. I

As $|\xi_2| \asymp \mu\gamma^\nu$, $\gamma \geq C_1(\mu^{-1}h)^{1/(\nu+1)}$ let us consider first eigenvalues $\Lambda_n(x_2, \xi_2)$ of operator $\mathcal{A}(x_2, \xi_2)$. Then proposition A.3 implies instantly that

(8.2) As $n \neq l$ and $|\xi_2| \asymp \mu\gamma^\nu$, $\gamma \geq C_1(\mu^{-1}h)^{1/(\nu+1)}$

$$\Lambda_n(x_2, \xi_2) \asymp (n - l)\mu h \gamma^{\nu-1}$$

and signs of the left and right-hand expressions coincide and

$$\Lambda_l(x_2, \xi_2) = \omega_l h^2 \gamma^{-2} - \frac{1}{2} W(x_2) + O\Bigl(h^2\gamma^{-1} + h^2(\mu^{-1}h)^2\gamma^{-4-2\nu}\Bigr), \quad \omega_l > 0 \text{ as } l \geq 1. \tag{8.3}$$

Therefore

(8.4) As $l \geq 1$ zone $\mathcal{Y}''' \overset{\text{def}}{=} \Bigl\{C_0(\mu h^\nu)^{1/(\nu+1)} \leq |\xi_2| \leq \epsilon\mu h^\nu\Bigr\}$ is elliptic and its contribution to the remainder estimate is $O(h^s)$.

On the other hand,

(8.5) Under condition $(6.4)_\pm$ zone $\mathcal{Y}'' \stackrel{\mathrm{def}}{=} \bigl\{|\xi_2| \ge C\mu h^\nu\bigr\}$ is elliptic as well and its contribution to the remainder estimate is $O(h^s)$ as well for $l \ge 0$.

Therefore as $l \ge 1$ and condition $(6.4)_\pm$ is fulfilled, one needs to analyze only two remaining zones $\mathcal{X}_1 = \bigl\{\epsilon\rho_1 \le |\xi_2| \le C\rho_1\bigr\}$, $\rho_1 = \mu h^\nu$ and $\mathcal{X}_0 = \bigl\{|\xi_2| \le C_0\rho_0\bigr\}$, $\rho_0 = (\mu h^\nu)^{1/(\nu+1)}$.

In the zone $\mathcal{X}_1$ propagation speed with respect to x_2 is in average $\asymp \rho^{-1}$ (with $\rho = \rho_1$) due to proposition A.3 again and the propagation speed with respect to ξ_2 is in average $O(1)$ and therefore one can take

$$T_0 = Ch|\log h|, \qquad T_1 = \epsilon_1\rho_1 \tag{8.6}$$

and for $T \in [T_0, T_1]$ propagation on the energy levels $\tau \in [-\epsilon, \epsilon]$ which started in $B(0, \frac{1}{2})$ does not leave $B(0,1)$ but the shift with respect to x_2 is $\asymp \rho^{-1}T$ and it satisfies logarithmic uncertainty principle and thus the spectral trace is negligible.

Remark 8.1. One should be more careful as $\mu \ge h^{-M}$ with arbitrarily large M and use $\log\mu$ instead of $|\log h|$.

Therefore

$$|F_{t\to h^{-1}\tau}\bar{\chi}_T(t)(Qu)| \tag{8.7}$$

does not exceed $Ch^{-1}\rho T_0 = C\rho|\log h|$ where Q is a partition element corresponding to $\mathcal{X}_1$, $|\tau| \le \epsilon$. Therefore due to Tauberian arguments the contribution of this zone to the remainder is $O(h^{-1}T_0/T_1) = O(|\log h|)$. One can get rid off this superficial logarithmic factor both in the estimate of (8.7) and in the remainder estimate; standard details I leave to the reader. So,

Proposition 8.2. *Let $l \ge 1$ and conditions $(6.4)_\pm$ and (8.1) be fulfilled. Then as Q is supported in the zone $\mathcal{X}_1$ expression (8.7) does not exceed $C\rho_1$ and the contribution of $\mathcal{X}_1$ to the remainder estimate is $O(1)$.*

Therefore I am left with the zone $\mathcal{X}_0 = \bigl\{|\xi_2| \le C_0(\mu h^\nu)^{1/(\nu+1)}\bigr\}$. Let us fix x_2. I don't know if eigenvalue $\lambda_n(\xi_2)$ of $\mathbf{a}^0(\xi_2)$ vanishes in $\mathcal{X}_0$ (may be even with some of its derivatives)[10] but I know that if it happens then $n \le c_1$; moreover due to the analyticity of $\lambda_n(\xi_2)$ it can happen only in no more then C_1 points and due to proposition A.3 and the analyticity of $\lambda_n(\xi_2)$

$$\lambda_n(\eta) \sim \alpha(\eta - \bar{\eta})^r \tag{8.8}$$

[10] It clearly happens for even ν and $n < l$.

for some $\alpha \ne 0$ and $r = 1, 2, \dots$ near each such point $\bar\eta$, α and r depend on $\bar\eta = \bar\eta_{n,k}$ $k = 1, \dots, K$ (depending on ν, l as well). Further, two eigenvalues do not vanish simultaneously.

But then condition $(6.4)_\pm$ will provide non-degeneracy. Really, in our assumptions an ellipticity is broken only in the strips of the type

$$\mathcal{Y} = \bigl\{ |\xi_2 - \bar\eta\rho_0| \asymp C\Delta \bigr\}, \qquad \Delta = \rho_0^{1-2/r}, \tag{8.9}$$

and the average propagation speed with respect to x_2 is of magnitude $\rho_0^{-1}|\xi_2 - \bar\eta|^{r-1} \asymp \rho_0^{(2-r)/r}$ there and therefore one can take

$$T_1 = \epsilon\rho_0^{-(2-r)/r}, \qquad T_0 = Ch|\log h|\rho_0^{-(2-r)/r}\Delta^{-1} \asymp h|\log h|, \qquad \Delta = \rho_0^{1-2/r}. \tag{8.10}$$

Therefore for Q supported in the strip $\mathcal{Y}$ expression (8.7) does not exceed $Ch^{-1}\Delta \times T_0 = C|\log h|\rho_0^{-(2-r)/r}$ and contribution of $\mathcal{Y}$ to the remainder estimate does not exceed this expression multiplied by T_1^{-1} i.e. $Ch|\log h|$. Furthermore, using standard methods one can easily get rid off the superficial logarithmic factor both in the estimate of (8.7) and the remainder estimate:

Proposition 8.3. *Let $l \ge 1$ and conditions $(6.4)_\pm$ and (8.1) be fulfilled. Then as Q is supported in the strip $\mathcal{Y}$ described by (8.9), expression (8.7) does not exceed $C\rho_0^{-(2-r)/r}$ and the contribution of $\mathcal{Y}$ to the remainder estimate is $O(1)$.*

Therefore I arrive to

Proposition 8.4. *Let $l \ge 1$ and conditions $(6.4)_\pm$ and (8.1) be fulfilled. Then the remainder estimate is $O(1)$ while the principal part is given by (6.40) for different strips with any $T \in [T_0, T_1]$ defined by (8.10) for strip $\mathcal{Y}$ under conditions (8.8) – (8.9) and by (8.6) for strip $\mathcal{X}_1$.*

I would like to note that

Proposition 8.5. *Let $l \ge 1$ and conditions $(6.4)_-$ and (8.1) be fulfilled. Then*
(i) Zone $\mathcal{X}_1$ is elliptic and its contribution to the remainder estimate is $O(h^s)$;
(ii) Furthermore if also condition

$$\lambda_n(\eta) \ne 0 \qquad \forall n, \eta \tag{8.11}$$

is fulfilled[11] *then the remainder estimate is $O(h^s)$.*

[11] However I cannot check condition (8.11).

8.2 Estimates. II

Let us consider the special case $l = 0$; I remind that then only eigenvalue $\lambda_0(\eta)$ should be considered and that condition $(6.4)_-$ leads then to the asymptotics with the principal part 0 and remainder estimate $O(h^s)$ and therefore is excluded from the further consideration.

Further, as ν is odd $\lambda_0 = 0$ identically, condition $(6.4)_+$ provides ellipticity everywhere. Thus I arrive to

Proposition 8.6. *Let $l = 0$, ν be odd and conditions $(6.4)_+$ and (8.1) be fulfilled. Then the remainder estimate is $O(h^s)$ while the principal part is given by (6.40).*

On the other hand, as $l = 0$, ν is even and condition $(6.4)_+$ holds due to proposition A.7 ellipticity is violated only in the strip

$$\mathcal{Y} = \{\epsilon_1\Delta \leq |\xi_2 - \eta\rho_0| \leq C\Delta\}, \qquad \eta \asymp |\log\rho_0|^{\nu/(\nu+1)},\ \Delta = \rho_0|\log\rho_0|^{-1/(\nu+1)} \tag{8.12}$$

where as before $\rho_0 = (\mu h^\nu)^{1/(\nu+1)}$. In this strip propagation speed with respect to x_2 is $\asymp \Delta^{-1}$ and again

$$T_0 = Ch|\log h|, \qquad T_1 = \epsilon\Delta \tag{8.13}$$

and expression (8.7) does not exceed $Ch^{-1}\Delta T_0 = C\Delta|\log h|$ and the remainder estimate is $O(|\log h|)$. Further, by the standard arguments one can get rid off the superficial logarithmic factors. Thus

Proposition 8.7. *Let $l = 0$, ν be even and conditions $(6.4)_+$ and (8.1) be fulfilled. Then the remainder estimate is $O(1)$ while the principal part is given by (6.40) with T_0, T_1 defined by (8.13).*

8.3 Estimates. III

Now I want to derive estimates under condition $(6.4)_\pm$ replaced by (6.5). Without condition $(6.4)_\pm$ some zones cease to be elliptic and should be reexamined:

(8.14) As $l \geq 1$ these zones are $\bigl\{|\xi_2| \geq C\mu h^\nu\bigr\}$ and also

(8.15) As $l \geq 1$ these zones are "inner parts" of the strips described by (8.9), namely, $\mathcal{Y} = \bigl\{|\xi_2 - \bar\eta\rho_0| \leq \epsilon_1\Delta\bigr\}$.

(8.16) As $l = 0$, ν even this zone is $\bigl\{|\xi_2| \geq C\rho_0|\log\rho_0|^{\nu/(\nu+1)}\bigr\}$;

(8.17) As $l = 0$, ν odd this zone is $\bigl\{|\xi_2| \leq \epsilon\mu\bigr\}$.

Since condition (6.5) provides $T_0 = Ch|\log h|$ anyway contribution of (8.9)-type strips to the remainder estimate will be $O(1)$ again. The standard partition-rescaling arguments in all other zones bring contribution of all other zones to $O(\log\mu)$; however additional arguments of the proof of proposition 6.14 allow us to reduce it to $O(1)$. Therefore

Proposition 8.8. *Let conditions* (*8.1*) *and* (*6.5*) *be fulfilled. Then the remainder estimate is* $O(1)$ *while the principal part of the asymptotics is given by* (*6.40*) *for different zones with any* $T \in [T_0, T_1]$, $T_0 = Ch|\log h|$ *and* T_1 *defined as in propositions 8.2–8.7.*

8.4 Calculations. I

In this subsection I give the principal parts of asymptotics already derived under condition $(6.4)_\pm$ in more explicit form.

First of all, consider method of successive approximations fixing $x_2 = y_2$. Then while contribution of the strip of the width Δ in ξ_2 to the principal part is of magnitude Δh^{-1}, each next term of successive approximations acquires factor $|\partial_{\xi_2}\Lambda_n| T \times T/h \asymp (\partial_{\xi_2}\Lambda_n)h|\log h|^2$ with $T = T_0$ where Λ_n is an eigenvalue of $\mathcal{A}$. Further one needs to consider only strips where ellipticity fails and then $\Delta \asymp |\partial_{\xi_2}\Lambda_n|^{-1}$.

So, the first, the second and the the third terms do not exceed

$$Ch^{-1}|\partial_{\xi_2}\Lambda_n|^{-1}, \quad C|\log h|^2, \quad Ch|\partial_{\xi_2}\Lambda_n|\cdot|\log h|^4 \tag*{$(8.18)_{1-3}$}$$

respectively.

Actually the second term in the successive approximations is $O(1)$. Really, considering the second term which corresponds to the linear part $(x_2 - y_2)\partial_{y_2}\mathcal{A}(y_2, hD_2)$ of the perturbation one can rewrite it as the result of direct calculations in the form including $\partial_{x_2}\partial_{\xi_2}\Lambda_n = 0$; on the other hand considering the second term corresponding to the rest $(x_2 - y)^2\mathcal{B}(x_2, y_2, hD_2)$ of the perturbation one can estimate it easily by $O(h^\delta)$.

Now I can rewrite the principal part of the asymptotics as

$$(2\pi h)^{-1}\int \mathrm{e}(x_1, x_1, 0; x_2, \xi_2)\psi(x)\, d\xi_2 dx \tag{8.19}$$

with error not exceeding already achieved remainder estimate which is either $O(1)$ or $O(h^\infty)$ (where remainder estimate $O(h^\infty)$ corresponds to the elliptic case and no successive approximations are needed at all).

Let us consider the contribution of the strips where ellipticity is broken to the error; I remind it does not exceed the minimum of all three expressions in $(8.18)_{1-3}$. Then $(8.18)_3$

is obviously $O(1)$ in all cases with the singular exception of the strip (8.9) with $r = 1$, $\rho h \ge |\log h|^{-K}$. However in this case $(8.18)_1$ is $O(1)$ unless $|\log h|^{-K} \le \rho h \le |\log h|^K$ and one can still handle this case getting rid off the superficial logarithmic factors in $(8.18)_{1,3}$ by the standard arguments. Thus I arrive to

Theorem 8.9. *Let conditions* $(6.4)_\pm$ *and* (*8.6*) *be fulfilled. Then*

(i) Asymptotics with the principal part given by (*8.19*) *holds with the remainder estimate* $O(1)$*;*

(ii) Furthermore, as $l = 0$, ν *is odd this asymptotics holds with the remainder estimate* $O(h^\infty)$.

Furthermore, fixing W at $x_1 = 0$ and $\alpha = 1$ and thus replacing $\mathcal{A}$ by $\mathcal{A}^0$ to the pilot model operator, I can apply the method of successive approximation again; then each next term gets an extra factor $C\gamma T_0 h^{-1}|\log h|$ with $\gamma = (\mu^{-1}|\xi_2|)^{1/\nu}$ and only strips where ellipticity breaks should be counted. Also one can see easily that

(8.20) The error does not exceed the second term $Ch^{-2}T_0\Delta\gamma$ [12]. Furthermore, for odd ν and perturbation, which is odd with respect to x_1, the second term is 0 and therefore the error does not exceed the sum of the second term with a perturbation $O(x_1^2)$ and the third term with a perturbation $O(x_1)$ i.e. $Ch^{-3}T_0^2\Delta\gamma^2$ [12].

Thus, I just list the different cases:

(8.21) As $l \ge 1$ and condition $(6.4)_+$ is fulfilled the main contribution to the error is provided by the zone $\mathcal{X}_1$ with $\xi_2 \asymp \mu h^\nu$ and $\gamma \asymp h$ and of the width $\Delta \asymp \mu h^\nu$; so the error is $O\bigl(\mu h^\nu\bigr)$. The contributions of (8.9)-type strips are much smaller;

(8.22) As $l \ge 1$ and condition $(6.4)_-$ is fulfilled the main contribution to the error is provided by (8.9)-type strips with the largest possible r; then $\xi_2 = O\bigl((\mu h^\nu)^{1/(\nu+1)}\bigr)$, $\gamma \asymp (\mu^{-1}h)^{1/(\nu+1)}$ and $\Delta \asymp (\mu h^\nu)^{(r-2)/r(\nu+1)}$; so the error is $O\bigl((\mu h^\nu)^{-\delta}\bigr)$ with $\delta = 2/r(\nu+1)$ anyway;

(8.23) As $l = 0$, ν is even and condition $(6.4)_+$ is fulfilled the main contribution to the error is provided by $\mathcal{X}_1$ with $\xi_2 \asymp (\mu h^\nu)^{1/(\nu+1)}|\log(\mu h^\nu)|^{\nu/(\nu+1)}$, $\gamma \asymp (\mu^{-1}h)^{1/(\nu+1)}|\log(\mu h^\nu)|^{1/(\nu+1)}$ and of the width $\Delta \asymp (\mu h^\nu)^{1/(\nu+1)}|\log(\mu h^\nu)|^{-1/(\nu+1)}$; so the error is $O(1)$ anyway;

[12] I skip superficial logarithmic factors one can easily get rid off by the standard arguments.

(8.24) As $l=0$, ν is odd and condition $(6.4)_+$ is fulfilled the error is just $O(h^\infty)$.

Thus I arrive to asymptotics with the principal part

$$(2\pi h)^{-1}\int \mathrm{e}_0(x_1,x_1,0;x_2,\xi_2)\psi(x)\,d\xi_2 dx \tag{8.25}$$

and remainder estimates described in Theorem 8.10 below:

Theorem 8.10. *Let condition (8.1) be fulfilled. Then*

(i) As $l\ge 1$ and condition $(6.4)_+$ is fulfilled asymptotics with the principal part given by (8.25) holds with the remainder estimate $O(\mu h^\nu)$;

(ii) As <u>either</u> $l\ge 1$ and condition $(6.4)_-$ is fulfilled <u>or</u> $l=0$, ν is even and condition $(6.4)_+$ is fulfilled asymptotics with the principal part given by (8.25) holds with the remainder estimate $O(1)$;

(iii) Furthermore, as $l=0$, ν is odd and condition $(6.4)_+$ is fulfilled the same asymptotics holds with the remainder estimate $O(h^\infty)$.

8.5 Calculations. II

In this subsection I give in more explicit form the principal parts of asymptotics already derived under condition (6.5). Basically I need to reconsider only the external formerly elliptic zones described by (8.14)–(8.17). The analysis in the first of them is not different from the analysis under condition $(6.4)_\pm$; analysis in the second one repeats the proof of theorem 6.17; analysis in two latter is rather obvious. Thus I arrive to two following theorems:

Theorem 8.11. *Let conditions (6.5) and (8.6) be fulfilled. Then asymptotics with the principal part (8.19) holds with the remainder estimate $O(1)$.*

Theorem 8.12. *Let conditions (6.5) and (8.6) be fulfilled. Then*

(i) As $l\ge 1$ estimate

$$\begin{aligned}\mathcal{R}^{**}\stackrel{\mathrm{def}}{=}|\int\Bigl(&e(x,x,0)-(2\pi h)^{-1}\int \mathrm{e}_0(x_1,x_1,0;x_2,\xi_2)\,d\xi_2\\ &-\mathcal{E}^{\mathrm{MW}}(x,0)+\mathcal{E}_0^{\mathrm{MW}}(x,0)\Bigr)\psi(x)\,dx-\sum\kappa_{l,m}\mu h^{1+2m}|\le C\mu h^\nu\end{aligned}\tag{8.26}$$

holds;

(ii) As $l=0$ estimate

$$|\int \Big(e(x,x,0) - (2\pi h)^{-1}\int \mathrm{e}_0(x_1,x_1,0;x_2,\xi_2)\,d\xi_2 - \mathcal{E}^{\mathrm{MW}}(x,0) + \mathcal{E}_0^{\mathrm{MW}}(x,0)\Big)\psi(x)\,dx| \le C \tag{8.27}$$

holds.

A Appendix: Eigenvalues of 1D operators

A.1 General observations

In this Appendix $\lambda_n(\eta)$ $(n=0,1,\dots)$ denote eigenvalues of one-dimensional pilot-model Schrödinger operators with $\mu=h=1$

$$\mathbf{a}^0(\eta) = D^2 + (\eta - x^\nu/\nu)^2 - (2l+1)x^{\nu-1} \tag{A.1}$$

or more general operator

$$\mathbf{a}(\eta) = (1+\alpha_1 x + \beta_1^2 x^2)D^2 + (1+\alpha_2 x + \beta_2^2 x^2)(\eta - x^\nu/\nu)^2 - (2l+1)(1+\alpha_3 x)x^{\nu-1} \tag{A.2}$$

with $\nu = 2,3,\dots$ and $\beta_j > \alpha_j^2/2$.

One can prove easily the following statement:

Proposition A.1. *Let $l\in\mathbb{R}$. Then*

(i) As $|\eta|\le C_0$ the spacing between two consecutive eigenvalues λ_n and λ_{n+1} with $n\le c_0$ is $\asymp 1$;

(ii) For operator (A.1) with odd ν $\lambda_n(-\eta)=\lambda_n(\eta)$;

(iii) For even ν and $\eta\le 0$ $\lambda_n(\eta)\ge (1-\epsilon)\eta^2 - C_1 \quad \forall n=0,1,\dots$.

However, the case of even ν and $\eta\to-\infty$ is rather exceptional:

Proposition A.2. *As $\eta\ge C_0$ (and thus also as $\eta\le -C_0$ and ν is odd)*

(i) The spacing between eigenvalues with $n\le c_0$ is $\asymp (1+|\eta|)^{(\nu-1)/\nu}$;

(ii) As $n<l$ $(l<n\le c_0)$ $\lambda_n(\eta)$ is less than (greater than respectively) $\epsilon(n-l)(1+|\eta|)^{(\nu-1)/\nu}$[13]*.*

Proof. Proof follows from the proof of proposition A.3 below. □

[13] Thus leaving the special case $n=l\in\mathbb{Z}^+$ for the further analysis.

A.2 Asymptotic behavior of $\lambda_l(\eta)$ as $\eta\to\infty$ as $l\geq 1$

In this subsection I prove

Proposition A.3. *(i) For operator (A.1) with $l\geq 1$ as $\eta\to+\infty$ (and thus also as $\eta\to-\infty$ and ν is odd)*

$$\lambda_l(\eta)=\kappa\eta^{-2/\nu}+O\bigl(\eta^{-(\nu+3)/\nu}\bigr) \tag{A.3}$$

with $\kappa>0$;
(ii) For operator (A.2) with $l\geq 1$ as $\eta\to+\infty$ (and thus as $\eta\to-\infty$ and ν is odd)

$$\partial_{\alpha_j}\lambda_l(\eta)\bigr|_{\alpha=\beta=0}=\kappa_j\eta+O(\eta^{-1/\nu}) \tag{A.4}$$

with $\kappa_1=\kappa_2=-\kappa_3/2$, $\alpha=(\alpha_1,\alpha_2,\alpha_3)$, $\beta=(\beta_1,\beta_2,\beta_3)$ and furthermore

$$\sum_{1\leq j\leq 3}\partial_{\alpha_j}\lambda_l(\eta)\bigr|_{\alpha=\beta=0}=\kappa_4\eta^{1/\nu}\lambda_l+O(\eta^{-2/\nu}). \tag{A.5}$$

Proof. (i) Let us plug $\eta=\gamma^\nu/\nu$ with $\gamma\gg 1$ where in the case even ν this is the only scenario and in the case of odd ν analysis of scenario $\xi_2=-\gamma^\nu/\nu$ is done by the symmetry. Then after shift $x\mapsto x+\gamma$ operator $\mathbf{a}^0(\eta)$ is transformed into operator

$$D^2+x^2\Bigl(\gamma^{\nu-1}+\frac{1}{2}(\nu-1)x\gamma^{\nu-2}+\frac{1}{6}(\nu-1)(\nu-2)x^2\gamma^{\nu-3}+\dots\Bigr)^2\\ -(2l+1)\Bigl(\gamma^{\nu-1}+(\nu-1)x\gamma^{\nu-2}+\frac{1}{2}(\nu-1)(\nu-2)x^2\gamma^{\nu-3}+\dots\Bigr)$$

and after rescaling $x\mapsto x\gamma^{(1-\nu)/2}$ this operator is transformed into $\gamma^{\nu-1}\mathbf{b}_\varepsilon$ where

$$\mathbf{b}_\varepsilon=D^2+x^2\Bigl(1+\frac{1}{2}(\nu-1)x\varepsilon+\frac{1}{6}(\nu-1)(\nu-2)x^2\varepsilon^2+\dots\Bigr)^2\\ -(2l+1)\Bigl(1+(\nu-1)x\varepsilon+\frac{1}{2}(\nu-1)(\nu-2)x^2\varepsilon^2+\dots\Bigr)$$

with $\varepsilon=\gamma^{-(\nu+1)/2}$. Then

$$\mathbf{b}_\varepsilon=\underbrace{D^2+x^2-(2l+1)}_{\mathbf{h}_0}+\varepsilon\underbrace{(\nu-1)\Bigl(x^3-(2l+1)x\Bigr)}_{\mathbf{h}_1}+\\ \varepsilon^2\underbrace{(\nu-1)\Bigl((\frac{7}{12}\nu-\frac{11}{12})x^4-\frac{1}{2}(2l+1)(\nu-2)x^2\Bigr)}_{\mathbf{h}_2}+O\bigl(\varepsilon^3\bigr)$$

and let us denote by Λ_ε and U_ε its eigenvalue close to 0 and the corresponding eigenfunction. Then

$$\Lambda_\varepsilon = \omega_1\varepsilon + \omega_2\varepsilon^2 + \dots \qquad \text{and} \qquad U_\varepsilon = u_0 + u_1\varepsilon + u_2\varepsilon^2 \dots \tag{A.6}$$

where obviously $u_0 = \upsilon_l$ is a Hermite function, $\omega_1 = \omega_3 = \dots = 0$ and

$$\mathbf{h}_0 u_1 + \mathbf{h}_1 u_0 = 0 \qquad \mathbf{h}_0 u_2 + \mathbf{h}_1 u + \mathbf{h}_2 u_0 = \omega_2 u_0. \tag{A.7}$$

Then

$$\omega_2 = \langle \mathbf{h}_1 u + \mathbf{h}_2 u_0, u_0\rangle = -\langle u, \mathbf{h}_0 u\rangle + \langle \mathbf{h}_2 u_0, u_0\rangle. \tag{A.8}$$

It is known that $(x - iD)\upsilon_k = (2k+2)^{1/2}\upsilon_{k+1}$, $(x + iD)\upsilon_k = (2k)^{1/2}\upsilon_{k-1}$ and therefore

$$\begin{aligned}
x\upsilon_l = & \frac{1}{2}\Big((2l+2)^{1/2}\upsilon_{l+1} + (2l)^{1/2}\upsilon_{l-1}\Big),\\
x^2\upsilon_l = & \frac{1}{4}\Big((2l+2)^{1/2}(2l+4)^{1/2}\upsilon_{l+2} + (4l+2)\upsilon_l + (2l)^{1/2}(2l-2)^{1/2}\upsilon_{l-2}\Big),\\
\big(x^2 - 2l - 1\big)\upsilon_l = & \frac{1}{4}\Big((2l+2)^{1/2}(2l+4)^{1/2}\upsilon_{l+2} - 2(2l+1)\upsilon_l + (2l)^{1/2}(2l-2)^{1/2}\upsilon_{l-2}\Big),\\
x\big(x^2 - 2l - 1\big)\upsilon_l = & \frac{1}{8}\Big((2l+2)^{1/2}(2l+4)^{1/2}(2l+6)^{1/2}\upsilon_{l+3} - (2l+2)^{1/2}(2l-2)\upsilon_{l+1} -\\
& (2l)^{1/2}(2l+4)\upsilon_{l-1} + (2l)^{1/2}(2l-2)^{1/2}(2l-4)^{1/2}\upsilon_{l-3}\Big),
\end{aligned}$$

which imply

$$\begin{aligned}
&\langle \mathbf{h}_0 u, u\rangle =\\
&\frac{1}{64}(\nu-1)^2\Big(\frac{1}{6}(2l+2)(2l+4)(2l+6)+\frac{1}{2}(2l+2)(2l-2)^2-\frac{1}{2}(2l)(2l+4)^2-\frac{1}{6}(2l)(2l-2)(2l-4)\Big) =\\
&\qquad\qquad \frac{1}{16}(\nu-1)^2\Big(-2l^2 - 2l + 3\Big).
\end{aligned}$$

On the other hand

$$\begin{aligned}
\langle \mathbf{h}_2 u_0, u_0\rangle = (\nu-1)\Big(&(\frac{7}{12}\nu - \frac{11}{12})\|x^2u_0\|^2 - \frac{1}{2}(\nu-2)(2l+1)\|xu_0\|^2\Big) =\\
&(\nu-1)(\frac{7}{12}\nu - \frac{11}{12})\cdot\frac{1}{16}\Big((2l+2)(2l+4) + (4l+2)^2 + (2l)(2l-2)\Big) -\\
&\frac{1}{4}(\nu-1)(\nu-2)\cdot(2l+1)^2 =\\
&(\nu-1)(7\nu-11)\cdot\frac{1}{16}\Big(2l^2+2l+1\Big) - \frac{1}{4}(\nu-1)(\nu-2)(2l+1)^2
\end{aligned}$$

and

$$\omega_2 = \frac{1}{16}(\nu-1)\Big((7\nu-11)\big(2l^2+2l+1\big)-4(\nu-2)\big(4l^2+4l+1\big)-(\nu-1)\big(-2l^2-2l+3\big)\Big) = \frac{1}{2}(\nu-1)l(l+1),$$

Therefore $\Lambda_\varepsilon = \omega_2\varepsilon^2 + O(\varepsilon^4)$ as $\varepsilon\to 0$ (because $\omega_3=0$ as well) which implies statement (i) with $\kappa=\omega_2\nu^{-2/\nu}$.
(ii) After obvious transformations

$$\partial_{\alpha_j}\lambda_l(\eta)\big|_{\alpha=\beta=0} = \gamma^{\nu-1}\langle \mathbf{k}_j U_\varepsilon, U_\varepsilon\rangle$$

with

$$\begin{aligned}
&\mathbf{k}_1 = (\gamma+\varepsilon x)D^2,\\
&\mathbf{k}_2 = x^2\Big(1+\frac{1}{2}(\nu-1)x\varepsilon+\frac{1}{6}(\nu-1)(\nu-2)x^2\varepsilon^2+\dots\Big)^2\\
&\mathbf{k}_3 = -(2l+1)\Big(1+(\nu-1)x\varepsilon+\frac{1}{2}(\nu-1)(\nu-2)x^2\varepsilon^2+\dots\Big)
\end{aligned}$$

and therefore

$$\langle \mathbf{k}_j U_\varepsilon, U_\varepsilon\rangle = \gamma\langle \mathbf{k}'_j u_0, u_0\rangle + O\big(\varepsilon^2\gamma\big)$$

with $\mathbf{k}'_1 = D^2$, $\mathbf{k}'_2 = x^2$, $\mathbf{k}'_3 = -(2l+1)$ which implies (A.4).

Known equalities $\langle x^2\upsilon_l,\upsilon_l\rangle = \langle D^2\upsilon_l,\upsilon_l\rangle = (2l+1)/2$ imply that $\kappa_1=\kappa_2=-\kappa_3/2$. Further, $\sum_{1\le j\le 3}\langle \mathbf{k}_j U_\varepsilon, U_\varepsilon\rangle = \gamma\lambda_l + O(\varepsilon^2)$ which implies (A.5). □

A.3 More general operators

Now I consider operator

$$\mathcal{A}(y,\eta) \stackrel{\mathrm{def}}{=} \beta\Big(\alpha h^2 D^2\alpha + \alpha^{-2}(\eta-\mu x^\nu/\nu)^2 - (2l+1)\mu h x^{\nu-1}\Big)\beta \tag{A.9}$$

with

$$\alpha=\alpha(x,y),\quad \beta=\beta(x,y),\quad \alpha(0,y)=1,\quad c_0^{-1}\le\beta\le c_0. \tag{A.10}$$

Let λ_n be eigenvalues of $\mathcal{A}$. Changing $x\mapsto\gamma(\mu^{-1}h)^{1/(\nu+1)}x$ and $\eta\mapsto(\mu h^\nu)^{1/(\nu+1)}$ respectively I arrive to operator (A.9) again with $\mu=h=1$ and α, β replaced by $\alpha((\mu^{-1}h)^{1/(\nu+1)}x,y)$, $\beta((\mu^{-1}h)^{1/(\nu+1)}x,y)$ and with a factor $(\mu h^\nu)^{2/(\nu+1)}$.

Proposition A.4. *Let conditions* (*A.9*), (*A.10*) *be fulfilled. Then*
(i) $\lambda_n(\eta)\ge C_0(\mu h^\nu)^{2/(\nu+1)}$ *as* $n\ge C$*;*
(ii) As $|\eta|\le C_0(\mu h^\nu)^{2/(\nu+1)}$ *the spacing between consecutive eigenvalues with* $n\le c_0$ *is* $\asymp (\mu h^\nu)^{2/(\nu+1)}$ *and*

$$|\partial_y^p\partial_\eta^q\lambda_n(y,\eta)|\le C_{pq}(\mu^{-1}h)^{p/(\nu+1)}(\mu h^\nu)^{(2-q)/(\nu+1)};\tag{A.11}$$

(iii) For even ν *and* $\eta\le 0$ $\quad\lambda_n(y,\eta)\ge(1-\epsilon)\eta^2-C_1$, $n=0,1,\dots$

Proposition A.5. *As* $\eta\ge C_0(\mu h^\nu)^{1/(\nu+1)}$ *(and thus also as* $\eta\le -C_0(\mu h^\nu)^{1/(\nu+1)}$ *and* ν *is odd)*
(i) The spacing between eigenvalues with $n\le c_0$ *is* $\asymp |\eta|^{(\nu-1)/\nu}(\mu h^\nu)^{1/\nu}$*;*
(ii) As $n<l$ $\;(l<n\le c_0)$ $\lambda_n(y,\eta)$ *is less than (greater than respectively)* $\epsilon(n-l)\bigl((\mu h^\nu)^{2/(\nu+1)}+|\eta|^{(\nu-1)/\nu}(\mu h^\nu)^{1/\nu}\bigr)$ *and these eigenvalues satisfy*

$$|\partial_y^p\partial_\eta^q\lambda_n(y,\eta)|\le C_{pq}(\mu^{-1}h)^{p/(\nu+1)}|\eta|^{-q}|\lambda_n(y,\eta)|;\tag{A.12}$$

(iii) As $\eta\ge C_0(\mu h^\nu)$ *(and thus as* $\eta\le -C_0(\mu h^\nu)$ *and* ν *is odd)* $|\lambda_l(y,\eta)|\le\epsilon_0$.

An extra analysis is needed for our purposes as $n=l$ and

$$\mu h^\nu\ge C_1\tag{A.13}$$

with arbitrarily large C_1.

Proposition A.6. *Let condition* (*A.13*) *be fulfilled and* $l\ge 1$*. Then as* $\eta\ge C_0(\mu h^\nu)^{1/(\nu+1)}$

$$\lambda_l(y,\eta)\asymp(\mu h^\nu/\eta)^{2/\nu}\qquad\textit{and}\qquad \eta\partial_\eta\lambda_l(y,\eta)\asymp(\mu h^\nu/\eta)^{2/\nu}.\tag{A.14}$$

A.4 Case of λ_l as $l=0$

Here cases of odd and even ν differ drastically. Note first that

$$\mathbf{a}^0(\eta)=\bigl(iD+\xi_2-x^\nu/\nu\bigr)\bigl(-iD+\xi_2-x^\nu/\nu\bigr)\tag{A.15}$$

and as ν is odd operator $\mathbf{a}^0(\eta)$ has the bottom eigenvalue $\lambda_0(\eta)$ with eigenfunction defined from $\bigl(-\partial+\xi_2-x^\nu/\nu\bigr)v=0$ i.e. $v=\exp(\xi_2x-x^{\nu+1}/\nu(\nu+1))$ and therefore $\lambda_0(\eta)$ is identically 0.

Similarly, as $\beta=1$ operator $\mathcal{A}$ defined by (A.9) is equal modulo $O(h^2)$ to operator

$$\mathcal{B}(y,\eta)\overset{\mathrm{def}}{=}h^2\alpha^2D+\alpha^{-2}(\eta-\mu x^\nu/\nu)^2-\mu h x^{\nu-1}= \bigl(ihD\alpha+\alpha^{-1}(\eta-\mu x^\nu/\nu)\bigr)\bigl(-\alpha ihD+\alpha^{-1}(\eta-\mu x^\nu/\nu)\bigr) \tag{A.16}$$

and I arrive to the statement (i) of

Proposition A.7. *(i) For odd ν the bottom eigenvalue of $\mathcal{B}(y,\eta)$ is 0;*
(ii) For even ν the bottom eigenvalue of $\mathcal{B}(y,\eta)$ is $(\mu h^\nu)^{2/(\nu+1)}\Lambda\bigl(y,\eta(\mu h^\nu)^{-1/(\nu+1)}\bigr)$ where

$$C^{-1}\exp(-C\eta^{(\nu+1)/\nu})\le\Lambda(y,\eta)\le C\exp(-\epsilon\eta^{(\nu+1)/\nu}), \tag{A.17}$$

$$\epsilon\eta^{1/\nu}\le-\partial_\eta\bigl(\log\Lambda(y,\eta)\bigr)\le C\eta^{1/\nu}. \tag{A.18}$$

Proof. I need to consider the case of even ν only. The same representation (A.15) shows that $\lambda_0(y,\eta)>0$. However, since this eigenfunction is fast decaying outside of the potential well, one can do the same shift and rescaling as before and using arguments of [HeMa] to prove that $\Lambda_0(y,\eta)\sim k\exp(-k_2\eta^{(\nu+1)/\nu})$. Also one can prove easily that $\partial_\eta\Lambda_0(y,\eta)\sim -k_3\eta^{1/\nu}k\exp(-k_2\eta^{(\nu+1)/\nu})$ as $\eta\ge C$ with $k_3=kk_2(1+\nu)/\nu$. Estimates (A.17), (A.18) follow from this. $\square$

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